

Dispersion relations and entropy of scalar fields in Rindler and de Sitter spaces

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Abstract

Properties of scalar fields in Rindler and de Sitter spaces are the subject of this work. Using the “brick wall model” the dispersion relations are determined and the remarkable properties common to both spaces as well as their differences are discussed. Equipped with these tools the horizon induced thermodynamics is revisited and shown to be dominated by a single mode propagating perpendicular to the horizon. Explicit expressions for the partition function, entropy and heat capacity for massless and massive fields are presented.

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I. INTRODUCTION

Dispersion relations are fundamental quantities in statistical physics. They determine the density of states and therefore the partition function and other thermodynamic quantities. Similarly, the dispersion relations of quantum fields in the presence of a Killing horizon determine the corresponding thermodynamic quantities. Before calculating them, singularities of the fields at the horizon have to be regularized. We adopt the most commonly used “brick wall” method [1–3] which regularizes these singularities by restricting the fields to a region close to but outside the horizon. The density of states of the quantum fields has been in general evaluated within the WKB approximation and it has been assumed that a large number of modes propagating perpendicular to the horizon contributes. Within this framework a variety of investigations (cf. the reviews [4], [5] where also other approaches are discussed) have been carried out and corrections to the semiclassical approximation have been investigated cf. [6], [7]. Other methods have been applied and related to the brick wall method such as the regularization via a Pauli-Villars method [8]. Although formulated originally in the context of a Schwarzschild black hole, this method applies to any space-time with a horizon as has been shown in [9].

We will determine the dispersion relations of a scalar field by imposing the brick wall boundary condition close to the horizon. Besides “exact” numerical results we will present analytical results in the small and large but partially overlapping energy regions. We will carry out this calculation separately for Rindler and de Sitter (static metric) spaces. Thereby we will establish quantitatively for both spaces the validity of the near horizon approximation which is straightforwardly generalized to any static spherically symmetric space like the Schwarzschild or Schwarzschild/AdS spaces. Using the results for the dispersion relations we will compute numerically the horizon induced thermodynamic properties, the partition function, entropy and heat capacity, for massless and massive scalar particles and provide accurate analytical results. It will be shown that finite mass effects are visible only under extreme conditions. Our results will be seen to differ by 1 to 2 orders of magnitude from the “standard” results of the brick wall model. The source of this discrepancy will be identified.

II. DISPERSION RELATIONS OF SCALAR FIELDS IN RINDLER AND DE SITTER SPACE

A. Dispersion relations in Rindler space

A uniformly accelerated observer in Minkowski space moves along the hyperbola [10]

$$x^2 - t^2 = \frac{1}{a^2}, \quad \mathbf{x}_\perp = 0.$$

with the acceleration denoted by a and the coordinates transverse to the motion by \mathbf{x}_\perp . After the coordinate transformation

$$t, x, x_\perp \rightarrow \tau, \xi, x_\perp : \quad t(\tau, \xi) = \frac{1}{a} e^{a\xi} \sinh a\tau, \quad x(\tau, \xi) = \frac{1}{a} e^{a\xi} \cosh a\tau, \quad (1)$$

a particle at rest in the observer’s system at $\xi = \xi_0$ corresponds to the uniformly accelerated motion in Minkowski space with acceleration $ae^{-a\xi_0}$. The space-time defined by Eq. (1) is

the Rindler space with the metric

$$ds^2 = e^{2\kappa\xi}(d\tau^2 - d\xi^2) - d\mathbf{x}_\perp^2, \quad \kappa = a, \quad (2)$$

where the identity of the acceleration a and the surface gravity κ has been used, cf. [11]. The coordinate transformation (1) is not one-to-one. The coordinates $-\infty < \tau, \xi < \infty$ cover only one quarter of the Minkowski space, the "Rindler wedge" R_+

$$R_\pm = \{x^\mu \mid |t| \leq \pm x\}.$$

Upon reversion of the sign of x in Eq. (1) it is the Rindler wedge R_- which is covered by the corresponding parametrization. No causal connection exists between the two Rindler wedges R_\pm .

We consider a non-interacting scalar field in Rindler space with the action

$$S = \frac{1}{2} \int d\tau d\xi d\mathbf{x}_\perp \{ (\partial_\tau \Phi)^2 - (\partial_\xi \Phi)^2 - (m^2 \Phi^2 + (\nabla_\perp \Phi)^2) e^{2\kappa\xi} \}. \quad (3)$$

The solutions of the equations of motion read

$$\Phi(\xi, \mathbf{x}_\perp) = e^{-i\omega\tau} e^{i\mathbf{k}_\perp \mathbf{x}_\perp} k_{i\omega/\kappa}(z(\xi)), \quad z(\xi) = \frac{1}{\kappa} \sqrt{m^2 + \mathbf{k}_\perp^2} e^{\kappa\xi}, \quad (4)$$

with the (appropriately normalized) MacDonald functions satisfying the differential equation

$$\left[-\frac{d^2}{d\xi^2} + (m^2 + \mathbf{k}_\perp^2) e^{2\xi} - \omega^2 \right] k_{i\omega/\kappa}(z(\xi)) = 0, \quad k_{i\omega/\kappa}(z) = \frac{1}{\pi} \sqrt{2\omega/\kappa \sinh \pi\omega/\kappa} K_{i\omega/\kappa}(z). \quad (5)$$

The quantization is standard, the restriction to the Rindler wedge R_+ however having severe consequences. Due to the elimination of the degrees of freedom in R_- which are correlated ("entangled") with the degrees of freedom in R_+ , [4, 12–15], a reduced density matrix appears and various quantities are affected like the average particle number in the Minkowski space vacuum [16],

$$\langle 0_M | a^\dagger(\omega, \mathbf{k}_\perp) a(\omega', \mathbf{k}'_\perp) | 0_M \rangle = \frac{1}{e^{2\pi\frac{\omega}{\kappa}} - 1} \delta(\omega - \omega') \delta(\mathbf{k}_\perp - \mathbf{k}'_\perp), \quad (6)$$

or the stationary propagator [17], (with η the proper distance between (ξ, \mathbf{x}_\perp) and $(\xi', \mathbf{x}'_\perp)$ in AdS_4 , i.e., the distance in a three-hyperboloid cf. [18]),

$$\int_{-\infty}^{\infty} d\tau e^{i\omega\tau} D(\tau, \xi, \xi', \mathbf{x}_\perp - \mathbf{x}'_\perp) \sim e^{i\frac{\omega\eta}{\kappa}} + \frac{2i \sin \frac{\omega\eta}{\kappa}}{e^{\frac{2\pi\omega}{\kappa}} - 1}, \quad (7)$$

with the second term being closely related to the Rindler space dispersion relations. The appearance of the factor $e^{2\pi\omega/\kappa} - 1$ suggests to use the language of statistical thermodynamics. We have to keep in mind that, unlike in thermodynamics proper, this quantity does not depend on Planck's constant \hbar . For this reason we use in the following $\tilde{\beta}$ with

$$\tilde{\beta}\omega = \frac{2\pi\omega}{\kappa} = \frac{\omega}{\tilde{T}}, \quad (8)$$

rather than multiplying numerator and denominator with \hbar and rewriting $\tilde{\beta}\omega = \hbar\omega/T_R$ with the Rindler temperature $T_R = \hbar\kappa/2\pi$. To simplify the notation we assume for the

following that dimensionful quantities are given in units of powers of κ and, possibly after differentiation, $\tilde{\beta}$ has to be identified with 2π .

For calculating the thermodynamic quantities, we follow the procedure in [2] and restrict the system under consideration to a part of the Rindler space. The resulting discrete spectrum consists of eigenvalues characterized by 3 integers n, n_2, n_3 and the basic thermodynamic quantity, the partition function, is given by,

$$\ln Z = - \sum_{n, n_2, n_3}^{\infty} \ln (1 - e^{-\tilde{\beta} \omega(n, n_2, n_3)}) .$$

In the transverse directions the system is restricted to a square with side-length \mathcal{L} . We impose periodic boundary conditions and replace the sum over n_2, n_3 by an integral,

$$\ln Z = - \frac{\mathcal{A}}{2\pi} \sum_{n=1}^{\infty} \int_0^{\infty} k_{\perp} dk_{\perp} \ln (1 - e^{-\tilde{\beta} \omega_n(k_{\perp})}) , \quad \mathcal{A} = \mathcal{L}^2 . \quad (9)$$

The restriction of the ξ variable has to account for the infinite degeneracy of the spectrum which arises, for $m = 0$, from the invariance of the action (3) under the transformation [19],

$$\phi(\tau, \xi, \mathbf{x}_{\perp}) \longrightarrow e^{\delta\xi} \phi(\tau, \xi', \mathbf{x}'_{\perp}) , \quad \xi' = \xi + \delta\xi , \quad \mathbf{x}'_{\perp} = e^{\delta\xi} \mathbf{x}_{\perp} ,$$

consisting of a translation in ξ and a rescaling of the perpendicular coordinates \mathbf{x}_{\perp} . The degeneracy persists also for massive fields though the underlying symmetry is more involved. There are various possibilities to remove the degeneracy. We also follow here the procedure in [2] and remove the degeneracy by requiring the space to be limited to the region $\xi \geq \xi_0$. We impose Dirichlet boundary conditions for the eigenmodes at the finite distance e^{ξ_0} from the horizon,

$$K_{i\omega}(\mathcal{K}) = 0 , \quad \text{with} \quad \mathcal{K} = e^{\xi_0} \sqrt{k_{\perp}^2 + m^2} . \quad (10)$$

Due to the exponential increase of the repulsive “potential” in the wave equation (5), a discrete spectrum with respect to the ξ variable is obtained without erecting a second wall. The appearance of the factor e^{ξ_0} is due to the unlimited increase of the time dilation when approaching the horizon as is seen explicitly for a classical particle moving in Rindler space with the Hamiltonian function given by,

$$H = \sqrt{\pi_{\xi}^2 + e^{2\xi}(m^2 + \pi_{\perp}^2)} .$$

It is straightforward to solve the equations of motion,

$$e^{2\xi} = \frac{e^{2\xi_0}}{\cosh^2 \tau} , \quad x_{\perp} = \frac{\pi_{\perp}}{E} e^{2\xi_0} \tanh \tau ,$$

with E denoting the energy of the particle,

$$E = e^{\xi_0} \sqrt{m^2 + \pi_{\perp}^2} = \mathcal{K}_{cpa} . \quad (11)$$

The transverse velocity tends exponentially to zero with $\xi_0 \rightarrow -\infty$. Therefore values of \mathcal{K} or \mathcal{K}_{cpa} of the same order of magnitude as ω can be reached only if the transverse momenta are large enough, i.e., $\pi_{\perp} \sim e^{-\xi_0}$, to compensate the effect of the time dilation. Although

derived for the Rindler space kinematics this argument applies, independently of the details, to static metrics exhibiting a non-extremal horizon.

In the following we denote with ℓ the distance of the boundary to the horizon e^{ξ_0} ,

$$\xi_0 = \ln \ell. \quad (12)$$

For evaluation of the thermodynamic quantities the level density (cf. Eqs. (9), (10)) associated with the transverse motion for each of the discrete levels implicitly given by Eq. (10) has to be computed. This can be achieved by considering the boundary condition (10) as an eigenvalue equation for $\mathcal{K}_n(\omega)$ at given ω with n denoting the number of zeroes of $\Phi(\xi, \mathbf{x}_\perp)$ for $\xi \geq \xi_0$. In terms of these functions the level density can be computed for any value of the parameters ξ_0 , \mathcal{L} , m ,

$$\frac{\mathcal{A}}{2\pi} \sum_{n=1}^{\infty} k_\perp dk_\perp = \sum_{n=1}^{\infty} \mu(\omega, n) d\omega, \quad \mu(\omega, n) = \frac{\mathcal{A}}{2\pi\ell^2} \mathcal{K}_n(\omega) \frac{d\mathcal{K}_n(\omega)}{d\omega} \theta(\mathcal{K}_n(\omega) - e^{\xi_0} m), \quad (13)$$

and yields the following expression for the logarithm of the partition function (cf. Eq. (12)),

$$\ln Z = \mathcal{A} e^{-2\xi_0} \ln z(\tilde{\beta}, m_\ell) = \frac{\mathcal{A}}{4\ell^2} \sum_{n=1}^{\infty} \zeta_n(\tilde{\beta}, \omega_n^0), \quad m_\ell = m\ell = me^{\xi_0}, \quad (14)$$

where, after an integration by parts, the functions ζ_n are given by,

$$\zeta_n(\tilde{\beta}, \omega_n^0) = \frac{1}{\pi} \left(\mathcal{K}_n^2(\omega_n^0) \ln(1 - e^{-\tilde{\beta}\omega_n^0}) + \tilde{\beta} \int_{\omega_n^0}^{\infty} d\omega \phi_n(\omega, \tilde{\beta}) \right), \quad \phi_n(\omega, \tilde{\beta}) = \frac{\mathcal{K}_n^2(\omega)}{e^{\tilde{\beta}\omega} - 1}, \quad (15)$$

with the lower limit of the ω -integration (cf. Eq. (10)),

$$\mathcal{K}_n(\omega_n^0) = m_\ell. \quad (16)$$

For computing the thermodynamic quantities, the dispersion relations, i.e., the dependences of the “momenta” \mathcal{K}_n on the energy ω , have to be known. The core results of our studies of the dispersion relations are displayed in Fig. 1. In a log-log plot are shown the dispersion relations $\mathcal{K}_n(\omega)$ for various values of n obtained by solving Eq. (10) numerically and by applying two different but overlapping approximations, the “pole dominance” and the WKB approximation together with the accuracy of these approximation (right part of the Figure). The dispersion relations in the small \mathcal{K}_n/ω regime are determined by the asymptotics, $\xi_0 \ll -1$, of the MacDonald functions, cf. [20],

$$k_{i\omega}(\mathcal{K}_n) \approx -\sqrt{\frac{2}{\pi}} \sin(\omega\xi_0 - \delta), \quad e^{2i\delta} = \frac{\Gamma(1+i\omega)}{\Gamma(1-i\omega)} \left(\frac{e^{-\xi_0} \mathcal{K}_n(\omega)}{2} \right)^{-2i\omega}.$$

Thus, $k_{i\omega}(\mathcal{K}_n)$ vanishes if $\omega\xi_0 - \delta + n\pi = 0$, and we obtain,

$$\ln \mathcal{K}_n(\omega) \approx \ln \tilde{\mathcal{K}}_n(\omega) = (\arg \Gamma(1+i\omega) - n\pi)/\omega + \ln 2. \quad (17)$$

The positions of the singularities of \mathcal{K}_n in the complex ω -plane coincide with the positions of the poles of the Rindler space propagator Fourier transformed in time, cf. Eq. (7), and we shall refer to (17) as pole dominance (“PD”) approximation.

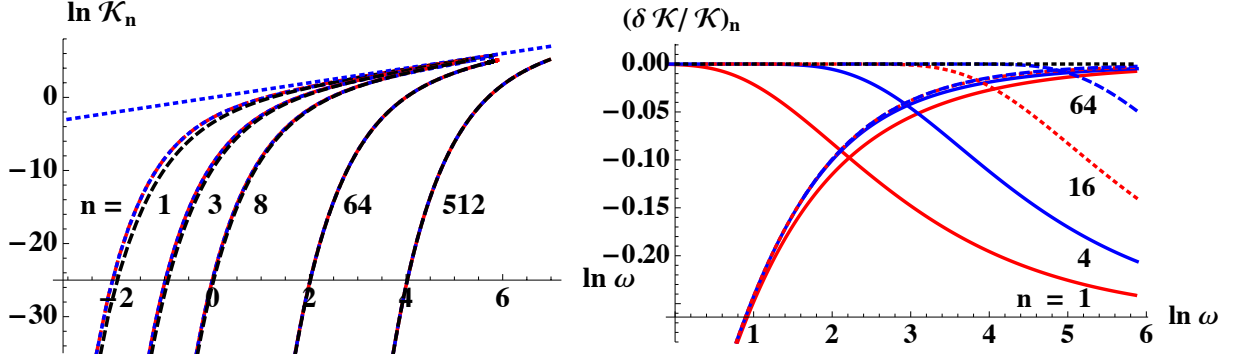


FIG. 1. Left: Double logarithmic plot of the dispersion relations for various values of n . Red dashed curves: Numerically determined $\mathcal{K}_n(\omega)$ (Eq. (10)), blue dashed curves: $\tilde{\mathcal{K}}_n(\omega)$, black dashed curves: $\hat{\mathcal{K}}_n(\omega)$, blue dotted curve: Dispersion relation of a classical particle in Rindler space, Eq. (11) with $\omega = E$, $\mathcal{K}_n = \mathcal{K}_{cpa}$. Right: Deviations of pole dominance (decreasing with $\ln \omega$) and WKB approximations (increasing) from the exact results $\tilde{\mathcal{K}}_n/\mathcal{K}_n - 1$ and $\hat{\mathcal{K}}_n/\mathcal{K}_n - 1$. The asymptotic value of the deviation $\tilde{\mathcal{K}}_n(\omega)/\mathcal{K}_n(\omega) - 1$ is given by the $\ln \omega$ -axis.

To analyze the shape of the dispersion relations we first consider the small ω region where the dispersion relation (17) simplifies,

$$\tilde{\mathcal{K}}_n \approx e^{-(n\pi/\omega + \gamma - \ln 2)}, \quad (18)$$

with Euler's constant γ . It exhibits an essential singularity at $\omega = 0$ which is responsible for the steep increase of \mathcal{K}_n with ω . This behavior has its origin in the infinite degeneracy of the spectrum of the Rindler space Hamiltonian in the absence of the boundary (cf. the wave equation (5)) corresponding to a vertical line for each value of ω . Furthermore, as in the degenerate case, in the regime of validity of Eq. (18), the curves for different n are parallel, i.e. for $\tilde{\mathcal{K}}_m(\omega_m) = \tilde{\mathcal{K}}_n(\omega_n)$ these curves satisfy $\ln \omega_m - \ln \omega_n = \ln m - \ln n$. The vertical distances between the curves on the left hand side of Fig. 1 are given by Eq. (17),

$$\ln \tilde{\mathcal{K}}_m(\omega) - \ln \tilde{\mathcal{K}}_n(\omega) = (n - m)\pi/\omega, \quad (19)$$

in agreement within 10% with the numerical results for $\omega \leq 8$ (cf. right hand side of Fig. 1). According to Eq. (17), the dispersion relations, evaluated in PD approximation, converge to the limit,

$$\tilde{\mathcal{K}}_n(\omega) \underset{\omega \rightarrow \infty}{\sim} \frac{2\omega}{e} e^{-(n-1/4)\pi/\omega},$$

which deviates by the factor $2/e$ from the asymptotics determined numerically.

The approach of the dispersion relations to that of a classical particle (11) suggests to apply for not too small values of ω , (cf. left part of Fig. 1), the WKB approximation [1] which has been the most common tool for calculating analytically thermodynamic quantities in Schwarzschild [1] and Rindler [2] spaces (cf. also the reviews [4] and [5]). The WKB dispersion relations, $\hat{\mathcal{K}}_n(\omega)$, are obtained by solving the equation, cf. [2], [1],

$$n\pi = \int_{\xi_0}^{\xi_1} d\xi \sqrt{\omega^2 - e^{2(\xi - \xi_0)} \hat{\mathcal{K}}_n^2},$$

where ξ_1 denotes the turning point. This integral can be evaluated analytically and the WKB dispersion relations are obtained as solutions of the following equation,

$$\frac{n\pi}{\omega} = h(\eta), \quad h(\eta) = -\sqrt{1-\eta^2} + \ln(\sqrt{1-\eta^2} + 1) - \ln \eta, \quad 0 \leq \eta = \frac{\hat{\mathcal{K}}_n}{\omega} \leq 1. \quad (20)$$

This equation implies that the solutions $\hat{\mathcal{K}}_n$ are given in terms of the inverse of the function,

$$\hat{\mathcal{K}}_n(\omega) = \omega h^{-1}\left(\frac{n\pi}{\omega}\right). \quad (21)$$

Expansion of $h(\eta)$ around $\eta = 0$ and $\eta = 1$ yields explicit expressions for the dispersion relations in the limit of small and of large ω ,

$$\hat{\mathcal{K}}_n(\omega) \underset{\omega \rightarrow 0}{\sim} \omega e^{-(n\pi/\omega + 1 - \ln 2)}, \quad \hat{\mathcal{K}}_n(\omega) \underset{\omega \rightarrow \infty}{\sim} \omega \left(1 + \frac{1}{2} \left(\frac{3n\pi}{\omega}\right)^{2/3} + \frac{7}{40} \left(\frac{3n\pi}{\omega}\right)^{4/3}\right)^{-1}.$$

As one might expect, the semiclassical WKB approximation fails to describe properly the dispersion relations in the small \mathcal{K}_n/ω limit (cf. Fig. 1). It reproduces however, in agreement with the numerical results (cf. Fig. 1), the $\omega \rightarrow \infty$ limit of the dispersion relation (11) of a classical particle at large ω . This limit is nevertheless rather subtle,

$$\hat{\mathcal{K}}_n(\omega) \frac{d\hat{\mathcal{K}}_n(\omega)}{d\omega} \underset{\omega \rightarrow \infty}{\sim} \omega - (2/3) (3n^2\pi^2)^{1/3} \omega^{1/3}. \quad (22)$$

Unlike in Minkowski space where $k dk/d\omega = \omega$, the subleading term of the density of states in Rindler space diverges with $\omega \rightarrow \infty$. A related consequence of the asymptotics of $\hat{\mathcal{K}}_n(\omega)$ is the increasing separation of the modes with different n ,

$$\hat{\mathcal{K}}_n(\omega) - \hat{\mathcal{K}}_m(\omega) \underset{\omega \rightarrow \infty}{\longrightarrow} \frac{(3\pi)^{2/3}}{2} (m^{2/3} - n^{2/3}) \omega^{1/3}. \quad (23)$$

This result will be important in our study of the thermodynamic properties of massive fields. For this application it is preferable to solve instead of Eq. (10) the equivalent equation,

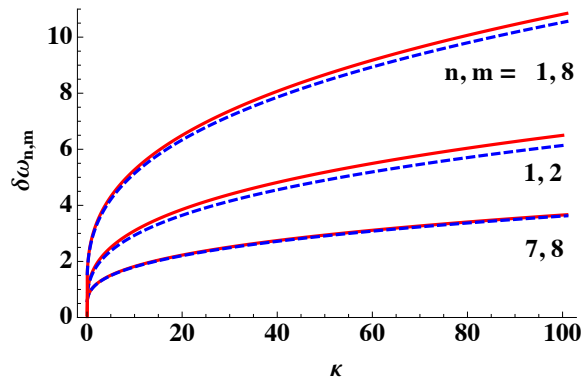


FIG. 2. Differences $\delta\omega_{n,m} = \omega_n(\mathcal{K}) - \omega_m(\mathcal{K})$. The difference $\delta\omega_{1,8}$ has been divided by a factor 3. Exact numerical (red) and WKB results (blue).

$k_{i\omega_n}(\mathcal{K}) = 0$, for arbitrary values of \mathcal{K} . Fig. 2 shows the effect of the mode separation and the divergence of the separation with increasing \mathcal{K} ,

$$\omega_n \underset{\mathcal{K} \rightarrow \infty}{\sim} \mathcal{K} + \frac{1}{2}(3n\pi)^{2/3} \mathcal{K}^{1/3}.$$

For neighboring curves and $n \gg 1$, we find to lowest order, in good agreement with the exact numerical results (cf. Fig. 2), $\delta\omega_{n+1,n} \approx (\pi^2\omega_n/3n)^{1/3}$.

B. Dispersion relations in de Sitter space

The purpose of the following study is twofold. We will first determine the dispersion relations in de Sitter space by applying analytical and numerical methods and at the same time we will establish quantitatively the connection between the Rindler and de Sitter space dispersion relations. Thereby we will exhibit quantitatively validity and limits of the near horizon approximation.

Starting point of our studies is the de Sitter space metric in static coordinates, cf. [21]

$$ds^2 = (1 - r^2\kappa^2)dt^2 - \frac{1}{1 - r^2\kappa^2}dr^2 - r^2d\Omega^2, \quad (24)$$

with the de Sitter radius given by the inverse of the surface gravity κ . Values of interest for cosmology or inflation are respectively

$$\kappa = 7.3 \cdot 10^{-27} \text{m}^{-1}, \quad \kappa\xi_0 = -140, \quad \text{and} \quad \kappa = 10^{34} \text{s}^{-1}, \quad \kappa\xi_0 = -20. \quad (25)$$

(As above all the quantities will be given in units of κ .) The radial eigenfunctions $\varphi_l(r)$ of the wave equation associated with the above metric are well known [22]. We impose the same type of boundary condition as for the Rindler space eigenfunctions,

$$\begin{aligned} \varphi_l(r) \Big|_{r^2=1-e^{2\xi_0}} &= 0, \\ \varphi_l(r) &= r^l(1-r^2)^{\frac{1}{2}i\omega} {}_2F_1\left((\mathcal{K}_+e^{-\xi_0} + i\omega)/2, (\mathcal{K}_-e^{-\xi_0} + i\omega)/2; l + 3/2; r^2\right), \end{aligned} \quad (26)$$

where we have introduced,

$$\mathcal{K}_{\pm} = e^{\xi_0} \left(l + 3/2 \pm \sqrt{9/4 - m^2} \right).$$

In accordance with Eq. (10), we have included in this definition the suppression factor e^{ξ_0} accounting for the time dilation of the transverse motion which, close to the horizon, affects the de Sitter and Rindler space dispersion relations in the same way. We therefore expect the relevant values of l to be of the order of $e^{-\xi_0}$.

By applying one of the linear transformation formulas for the hypergeometric function [23] the change in Eq. (26) from $1 - e^{2\xi_0}$ to the more appropriate variable $e^{2\xi_0}$ is achieved,

$$\varphi_l(\sqrt{1 - e^{2\xi_0}}) = (1 - e^{2\xi_0})^{l/2} \text{Re}[\rho(\omega, \mathcal{K}_+, \mathcal{K}_-, \xi_0) \sigma(\omega, \mathcal{K}_+, \mathcal{K}_-, \xi_0)], \quad (27)$$

with,

$$\begin{aligned} \rho(\omega, \mathcal{K}_+, \mathcal{K}_-, \xi_0) &= \frac{\Gamma((\mathcal{K}_+ + \mathcal{K}_-)e^{-\xi_0}/2) \Gamma(-i\omega) e^{i\omega\xi_0}}{\Gamma((\mathcal{K}_+e^{-\xi_0} - i\omega)/2) \Gamma((\mathcal{K}_-e^{-\xi_0} - i\omega)/2)}, \\ \sigma(\omega, \mathcal{K}_+, \mathcal{K}_-, \xi_0) &= {}_2F_1\left((\mathcal{K}_+e^{-\xi_0} + i\omega)/2, (\mathcal{K}_-e^{-\xi_0} + i\omega)/2; i\omega + 1; e^{2\xi_0}\right), \end{aligned} \quad (28)$$

and the boundary condition rewritten as,

$$\psi(\omega, \mathcal{K}_+, \mathcal{K}_-, \xi_0) = \arg \rho(\omega, \mathcal{K}_+, \mathcal{K}_-, \xi_0) + \arg \sigma(\omega, \mathcal{K}_+, \mathcal{K}_-, \xi_0) = -\left(n - \frac{1}{2}\right)\pi. \quad (29)$$

Solution of this equation which we have carried out numerically yields the de Sitter space dispersion relations. For analytical studies, this equation serves as starting point for the “near horizon approximation” which we define as the expansion in terms of the distance e^{ξ_0} to the horizon. To leading and next to leading order we obtain from Eq. (28), by treating l as a continuous variable,

$$\arg \rho_0 = \arg \Gamma(-i\omega) + \omega \ln(\mathcal{K}_{dS}/2), \quad \arg \rho_1 = -\omega \frac{\mathcal{K}_+ + \mathcal{K}_-}{2\mathcal{K}_+\mathcal{K}_-} e^{\xi_0} = -\omega \frac{l + 3/2}{l^2 + 3l + m^2} \quad (30)$$

and after a tedious calculation,

$$\arg(\sigma_0 + \sigma_1) \approx \arg \sigma_0 - \frac{\mathcal{K}_+ + \mathcal{K}_-}{2\mathcal{K}_{dS}} e^{\xi_0} \text{Im}\left(I_{i\omega+1}(\mathcal{K}_{dS})/I_{i\omega}(\mathcal{K}_{dS})\right), \quad (31)$$

where

$$\mathcal{K}_{dS} = \sqrt{\mathcal{K}_+\mathcal{K}_-} = e^{\xi_0} \sqrt{l(l+3) + m^2}. \quad (32)$$

For $\omega \leq 3$ the contribution of σ to ψ is negligible and the dispersion relations satisfy,

$$\arg \Gamma(-i\omega) + \omega \ln(\mathcal{K}_{dS}/2) = -\left(n - \frac{1}{2}\right)\pi.$$

These solutions coincide with the Rindler space dispersion relations obtained in the PD approximation (17), i.e., for given n and ω the identity, $\mathcal{K}_{dS}(n, \omega) = \mathcal{K}_n(\omega)$, holds. As in Rindler space, the appearance of $\Gamma(-i\omega)$ reflects the presence of poles (in the complex ω plane) of the de Sitter space propagator Fourier transformed in time.

To calculate σ_0 we order the terms in the hypergeometric function (28) according to powers of e^{ξ_0} and find to leading order in this expansion cf. [20],

$$\sigma_0(\omega, \mathcal{K}_+, \mathcal{K}_-, \xi_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\mathcal{K}_{dS}}{2}\right)^{2n} \frac{\Gamma(i\omega + 1)}{\Gamma(i\omega + n + 1)} = i\omega \Gamma(i\omega) (\mathcal{K}_{dS}/2)^{-i\omega} I_{i\omega}(\mathcal{K}_{dS}). \quad (33)$$

Combining with the leading order term ρ_0 (Eq. (30)), the approximate boundary condition (cf. (29)) reads,

$$\psi_0(\omega, \mathcal{K}_+, \mathcal{K}_-, \xi_0) = \arg I_{i\omega}(\mathcal{K}_{dS}) + \frac{\pi}{2} = -n\pi + \frac{\pi}{2}. \quad (34)$$

The relation [20],

$$K_{i\omega}(\mathcal{K}) = \frac{\pi}{\sinh \omega \pi} \text{Im} I_{i\omega}(\mathcal{K}),$$

implies that the zeroes of $K_{i\omega}(\mathcal{K})$ coincide with the zeroes of $\text{Im} I_{i\omega}(\mathcal{K})$. Thus to leading order the dispersion relations and therefore the thermodynamic quantities in de Sitter space coincide with the corresponding quantities in Rindler space with the identification of the “momenta” given in Eq. (32).

Origin as well as limitations of the connection between de Sitter and Rindler space dispersion relations are easily identified by comparing the corresponding wave equations. To

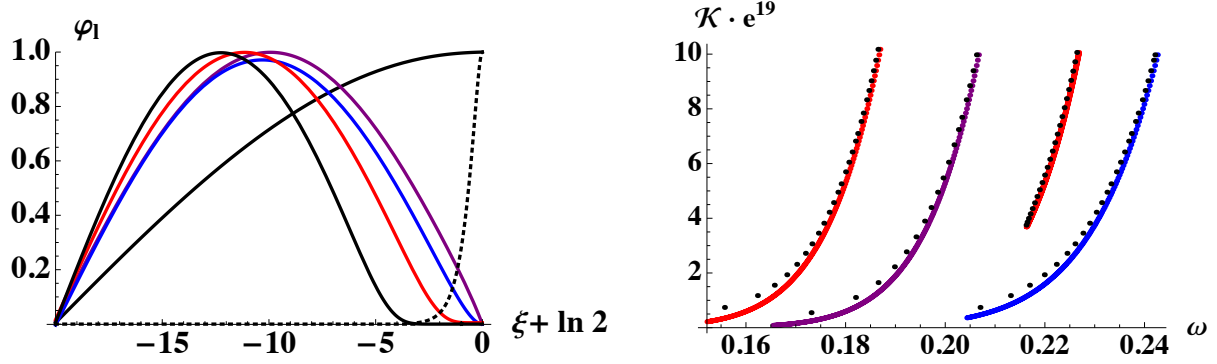


FIG. 3. Left: Absolute value of the de Sitter wave functions (26) as a function of ξ for $m = 0$, and $l = 0, 1, 2, 10, 100$ with the maxima normalized to 1. Dashed black curve: δV , Eq. (36). Right: de Sitter space dispersion relations (29) for $\xi_0 = -20$, $l \leq 26$ (black dots) in comparison with the Rindler space dispersion relations (solid lines) with $\mathcal{K}(\omega)$, Eq. (32).

this end we change the de Sitter space coordinate $r = -\tanh(\xi + \ln 2)$, $\xi \leq -\ln 2$, which yields the following wave equation,

$$\left[-\frac{d^2}{d\xi^2} + V_l(\xi + \ln 2) - \omega^2 \right] \varphi_l(\xi) = 0, \quad \text{with} \quad V_l(\xi) = \frac{l(l+1)}{\sinh^2 \xi} + \frac{m^2 - 2}{\cosh^2 \xi}. \quad (35)$$

Equation (35) can be replaced by the Rindler space wave equation (Eq. (5)) provided $\varphi_l(\xi)$ is localized sufficiently close to the horizon. For this to happen the centrifugal barrier has to be sufficiently large, $(l(l+1) + m^2 - 2)e^{2\xi} \gg \omega^2$. The tighter and tighter localization with increasing angular momentum is illustrated in Fig. 3. Also shown is the relative difference,

$$\delta V = (V_l(\xi) - 4(l(l+1) + m^2)e^{2\xi})/V_l(\xi), \quad (36)$$

for $l = 5$, $m = 0$.

Already for values with angular momenta as small as $l = 1$, only a weak overlap between the eigenfunction φ_1 and the difference δV between de Sitter and Rindler space “potentials” is found. On the other hand, due to the absence of the centrifugal barrier the wave-function for $m = 0$ and $l = 0$ is not dominated by the near horizon region.

These considerations lead us to consider in detail the dispersion relation at small angular momenta l where significant differences between de Sitter and Rindler space results occur. On the right hand side of Fig. 3 are shown the discrete eigenvalues for $n = 1$ and $m = 0, 1.5, 10$ (first 3 curves) and $n = 8$, $m = 0$ together with the corresponding Rindler space dispersion relations. The energies ω of the second and third dispersion relations are shifted by 0.02 and 0.04 respectively and reduced by a factor of 6 for the $n = 8$ dispersion relation. Not included in the figure are the eigenvalues for $l = 0$ with exact 0.0798, 0.119, 0.176, 1.18 and approximate values 0, 0.147, 0.176, 0. As suggested by the behavior of the wave functions, the de Sitter dispersion relations approach fast the Rindler space dispersion relations with increasing l and/or m . Already for $l = 1$, $m = 0$ Rindler and de Sitter space results (cf. Eq. (32)) agree within 4%. The discrepancy can be reduced to 1% by including the next to leading order (cf. Eqs. (30), (31)) in the near horizon approximation.

For $l = 0$, $m = 0$ (and $\omega \neq 0$), the near horizon approximation fails. The correspondence (32) assigns to $l = m = 0$ the Rindler space value $\tilde{\mathcal{K}} = \omega = 0$ (cf. Eq. (18)) independent of the value of n and the next to leading order (30) of the near horizon approximation diverges.

However, a closed expression for the dispersion relation can be obtained by applying the duplication formula for the Γ -functions in (28),

$$\omega(\xi_0 - \ln 2) + (1 - \delta_{l=0}) \sum_{l'=0}^{l-1} \arg(l' + i\omega) + \arg(l + 1 + i\omega) = (-n + 1/2) \pi. \quad (37)$$

For $l = 0$ and not too large values of n , the approximate dispersion relations are given by,

$$\omega_n = -\frac{n - 1/2}{\xi_0 + 1 - \ln 2} \pi, \quad (38)$$

which reproduces the exact values for $n \leq 8$ with an accuracy of 1% or better.

C. Dispersion relations in static spherically symmetric spaces close to horizons

The connection between the dispersion relations of scalar fields in de Sitter and in Rindler space can be generalized to a larger class of static spherically symmetric spaces with a non-extremal horizon. Besides the de Sitter space, the Schwarzschild, the Schwarzschild/AdS or the Reissner-Nordström space belong to this class. The common structure of the line element of this class of spaces reads cf. [5],

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2, \quad (39)$$

with the function $f(r)$ vanishing at $r = r_0$ and, close to the horizon, is approximately given by,

$$f(r) \approx (r - r_0) f'(r_0), \quad \text{with} \quad |(r - r_0) f'(r_0)| = e^{2\kappa\xi}, \quad \text{and} \quad 2\kappa = |f'(r_0)|. \quad (40)$$

The approximate metric (39) reads,

$$ds^2 \approx e^{2\kappa\xi} (dt^2 - d\xi^2) - r_0^2 \left(1 + \frac{1}{2\kappa r_0} e^{2\kappa\xi}\right)^2 d\Omega^2 \approx e^{2\kappa\xi} (dt^2 - d\xi^2) - r_0^2 d\Omega^2. \quad (41)$$

The last step of the approximation is valid only if the radial eigenfunctions are concentrated in the region close to the horizon which is never the case for vanishing angular momentum l and mass m . The approximate metric (41) is the metric of a product space of the 1 + 1 Rindler space and the 2-sphere. Comparison of this metric with the Rindler space metric (2) shows that (up to the normalization) the eigenfunctions are given by the MacDonald functions, which vanish at the boundary (cf. Eq. (10)),

$$K_{i\omega}(\mathcal{K}_{sp}) = 0, \quad \mathcal{K}_{sp} = e^{\xi_0} \sqrt{m^2 + l(l+1)/r_0^2}, \quad (42)$$

with ξ_0 , r_0 and m given in units of $1/\kappa$ and κ respectively. At this point we can proceed as above in identifying the dispersion relations of Rindler and spherical Rindler space.

To test the range of validity of this type of “near horizon approximation”, we apply the above approximation to de Sitter space where, according to Eq. (40),

$$r_0 = 1/\kappa, \quad (43)$$

and the near horizon metric and \mathcal{K}_{sp} are given by,

$$ds^2 = e^{2\xi}(dt^2 - d\xi^2) - d\Omega^2, \quad \mathcal{K}_{sp} = e^{\xi_0} \sqrt{m^2 + l(l+1)} = \left(\frac{m^2 + l(l+1)}{m^2 + l(l+3)} \right)^{1/2} \mathcal{K}_{dS}. \quad (44)$$

For vanishing m and $l > 0$, the two quantities \mathcal{K}_{dS} and \mathcal{K}_{sp} differ by up to 30% and approach each other with increasing l . Trivially at large l , but also at small l where the slope of ω as function of \mathcal{K} is of the order of 10^{-3} , the dispersion relations are only weakly affected, i.e., with the exception of the $l = 0, m = 0$ case, the dispersion relations are accurately described by the near horizon approximation (44).

Other examples where this method for calculating the dispersion relations and the thermodynamic quantities can be applied to are,

- Schwarzschild metric

$$f(r) = 1 - R_S/r, \quad R_S = 2GM, \quad \kappa = 1/2R_S, \quad \kappa r_0 = \frac{1}{2}, \quad (45)$$

- Schwarzschild/AdS metric

$$f(r) = 1 - \frac{R_S}{r} + \frac{r^2}{R^2}, \quad \kappa = \frac{1}{R b^2(\rho)} (\rho + b^3(\rho)), \quad \kappa r_0 = \frac{1}{b(\rho)} (\rho + b^3(\rho)), \quad (46)$$

$$\text{with } \rho = \frac{R_S}{2R}, \quad b(\rho) = \rho^{1/3} \left((\sqrt{1 + 1/27\rho^2} + 1)^{1/3} - (\sqrt{1 + 1/27\rho^2} - 1)^{1/3} \right),$$

- Reissner-Nordström metric

$$f(r) = 1 - \frac{R_S}{r} + \frac{R^2}{r^2}, \quad R = \ell_P Q, \quad \text{Planck length } \ell_P, \text{ charge } Q, \quad (47)$$

$$\rho = \sqrt{1 - 4R^2/R_S^2}, \quad \kappa = \frac{2\rho}{R_S(1+\rho)^2}, \quad \kappa r_0 = \frac{\rho}{1+\rho}.$$

In concluding this section we emphasize the universality of the dispersion relations (10). Having determined the Rindler space dispersion relations for a sufficiently large number of modes, as shown in Fig. 1, the (discrete) eigenvalues $\omega(n, l, m, r_0, \xi_0)$ for any static spherically symmetric space can be read off from this figure by identifying \mathcal{K} with \mathcal{K}_{sp} (Eq. (42)) and by taking into account that the scale of any dimensionful quantity is given by the appropriate power of the surface gravity. Evaluation of thermodynamic quantities requires summation over the angular momenta l and the number of zeroes n . If the distance to the horizon satisfies $e^{\xi_0} \ll 1$ significant contributions to the sum over angular momenta are obtained only if $l \gg 1$ and the summation can be replaced by integration over l , cf. Eq. (42), (the summation over n is not replaced by an integration),

$$(2l+1)dl = r_0^2 e^{-2\xi_0} d(\mathcal{K}^2).$$

Therefore the equations (13)-(16) apply with the area given by,

$$\mathcal{A} = \pi r_0^2, \quad (48)$$

while the quantities $\zeta_n(\tilde{\beta}, \omega_n^0)$ (Eq. (15)) are “universal”, i.e., independent of the parameters of the static, spherically symmetric metrics with a non-extremal horizon.

III. THERMODYNAMIC QUANTITIES

A. Thermodynamic properties of massless quantum fields

Given the dispersion relations, the thermodynamic quantities are, apart from a boundary term, (present only if $m \neq 0$) obtained by integrating $\mathcal{K}_n^2(\omega)$ over ω weighted with the “Boltzmann factor” (Eq. (15)) and summing over n . The integrands are shown in Fig. 4 for $n = 1, 2, 3$ and demonstrate dominance of the $n = 1$ contribution which in turn is dominated by the maximum at $\omega \approx 1$ with $\mathcal{K}_1 \approx 0.06$ (cf. Fig. 1). These universal values expressed in units of the surface gravity covers a large range in terms of “absolute” values, cf. Eqs. (10), (42),

$$\omega = \kappa, \quad \mathcal{K} = 0.06 \omega, \quad k_\perp = \mathcal{K} e^{-\xi_0}, \quad l = r_0 k_\perp,$$

as is demonstrated in Table I, cf. Eqs. (43), (45). We have determined numerically the loga-

	Rindler	Schwarzschild	de Sitter 1	de Sitter 2
$\hbar\omega[\text{MeV}]$	$3 \cdot 10^{-17}$	$3 \cdot 10^{-17}$	$1 \cdot 10^{-39}$	$7 \cdot 10^{12}$
$\hbar k_{\perp,1}$	$7 \cdot 10^{20} \text{ MeV}$	$1 \cdot 10^{37}$	$5 \cdot 10^{59}$	$1 \cdot 10^8$

TABLE I. Position ω of the maximum ϕ_1 (cf. Fig. 4) in units of MeV and the values of \mathcal{K}_1 in units of MeV for Rindler space and the values of the angular momenta for the Schwarzschild space with surface gravity of a black hole of one solar mass and for de Sitter space with the values of the surface gravity given in Eq. (25).

rithm of the partition function (Eq. (14)) and the entropy (cf. Eq. (15)),

$$S = \frac{\mathcal{A}}{4\ell^2} s(m_\ell), \quad s(m_\ell) = \left(1 - \tilde{\beta} \frac{\partial}{\partial \tilde{\beta}}\right) \ln z(\tilde{\beta}, m_\ell) \Big|_{\tilde{\beta}=2\pi}, \quad (49)$$

and have studied analytically these quantities in the PD and the WKB approximation. The n -sum can be carried out with the help of the relations (17) and (21),

$$\tilde{\mathcal{K}}_n(\omega) = \tilde{\mathcal{K}}_1(\omega) e^{(1-n)\pi/\omega}, \quad \hat{\mathcal{K}}_n(\omega) = n \hat{\mathcal{K}}_1(\omega/n), \quad (50)$$

and the calculation of the logarithm of the partition functions is reduced to quadratures,

$$\ln \tilde{Z} = \frac{\tilde{\beta} \mathcal{A}}{\pi \ell^2} \int_0^\infty \frac{e^{2 \arg \Gamma(1+i\omega)/\omega} d\omega}{(e^{\tilde{\beta}\omega} - 1)(e^{2\pi/\omega} - 1)}, \quad (51)$$

$$\ln \hat{Z} = \frac{\tilde{\beta} \mathcal{A}}{4\pi \ell^2} \sum_{n=1}^\infty n^3 \int_0^\infty d\omega \frac{1}{e^{\tilde{\beta} n \omega} - 1} \hat{\mathcal{K}}_n^2(\omega) \approx \frac{\tilde{\beta} \mathcal{A}}{4\pi \ell^2} \int_0^\infty d\omega e^{-\tilde{\beta}\omega} \frac{1 + 4e^{-\tilde{\beta}\omega} + e^{-2\tilde{\beta}\omega}}{(1 - e^{-\tilde{\beta}\omega})^4} \hat{\mathcal{K}}_1^2(\omega). \quad (52)$$

In the WKB approximation $\hat{\mathcal{K}}_1(\omega)$ is only known implicitly, cf. Eq. (20). The results of our studies of the thermodynamic quantities are compiled in Table II. Up to corrections of about 1%, the $n = 1$ terms of the logarithm of the partion function (and similarly of the entropy) coincide with the n -summed results reflecting the strong suppression of $\ln \Phi_n$ with increasing n as displayed in Fig. 4. The PD results agree with the corresponding numerically determined “exact” results with an accuracy of better than 1%, cf. Fig. 4. The order of magnitude of the

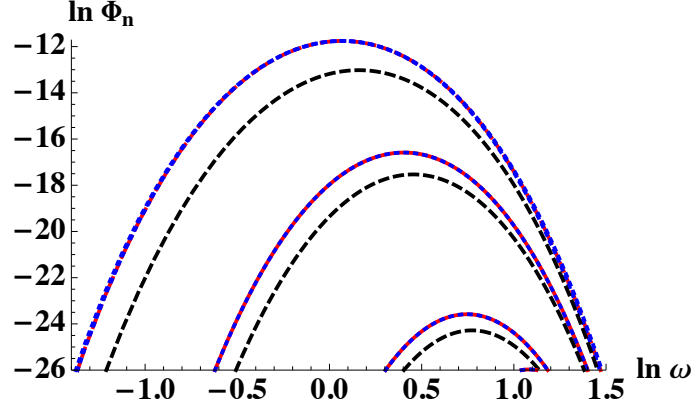


FIG. 4. Logarithm of the integrand ϕ_n of the partition function (Eq. (15)) as a function of ω for $n = 1, 2, 3$ the color code is the same as in Fig.1.

	ζ_1	$\sum_n \zeta_n$	$s(0)$
NUM	$1.27 \cdot 10^{-5}$	$1.30 \cdot 10^{-5}$	$9.68 \cdot 10^{-5}$
PD	$1.26 \cdot 10^{-5}$	$1.28 \cdot 10^{-5}$	$9.64 \cdot 10^{-5}$
WKB	$3.72 \cdot 10^{-6}$	$3.77 \cdot 10^{-6}$	$3.08 \cdot 10^{-5}$

TABLE II. $n = 1$ and summed contributions to the logarithm of the partition function and the entropy up to the factor $\mathcal{A}/4\ell^2$ obtained numerically, in the PD and WKB approximations.

entropy and its dependence on the size $\tilde{L} = 1/\kappa$ and “temperature” \tilde{T} , cf. Eq. (8), is easily estimated by evaluating the integral (51) in the saddle point approximation,

$$\begin{aligned}
S &\approx (1 + 2\pi) \ln \tilde{Z} \approx \frac{\mathcal{A}}{4\ell^2} 2(1 + 2\pi) \int_0^\infty e^{-\tilde{\beta}\omega - 2\pi\kappa/\omega + (2\kappa/\omega) \arg \Gamma(1+i\omega/\kappa)} d\omega/\kappa \\
&\approx \frac{\mathcal{A}}{\ell^2} (1 + 2\pi) \left(\frac{2}{\pi \tilde{T} \tilde{L}} \right)^{1/4} \cdot e^{(-4\pi + 2 \arg \Gamma(1+i\sqrt{2\pi \tilde{T} \tilde{L}})) / \sqrt{2\pi \tilde{T} \tilde{L}}} \underset{2\pi \tilde{T} \tilde{L}=1}{\approx} \frac{\mathcal{A}}{4\ell^2} 8.0 \cdot 10^{-5}. \quad (53)
\end{aligned}$$

which agrees within 20% with the numerical result in Table II. For comparison we consider the properties of the WKB results in Table II. As expected on the basis of the softer dispersion relations (cf. Fig. 1) with the reduced values of the transverse momenta for given ω (by a factor of 2 at $\omega = 1$) partition function and entropy are correspondingly suppressed relative to the exact and PD results by a factor of about 3. The WKB approximation has been employed in previous investigations with results for the entropy and partition function which exceed ours by two orders of magnitude. In order to identify the source of this discrepancy we employ the procedure underlying these calculations and replace in Eq. (52) the summation over n , (the index of the modes in Eq. (14)) by an integration,

$$\sum_{n=1}^{\infty} \frac{n^3}{e^{n\tilde{\beta}\omega} - 1} \rightarrow \int_0^\infty \frac{\nu^3}{e^{\nu\tilde{\beta}\omega} - 1} d\nu = \frac{1}{240} \left(\frac{2\pi}{\tilde{\beta}\omega} \right)^4. \quad (54)$$

The remaining integral (52) can also be carried out analytically (cf. Eqs. (20), (21)),

$$\int_0^\infty d\omega \frac{\hat{\mathcal{K}}_1^2(\omega)}{\omega^4} = -\frac{1}{\pi} \int_0^1 d\eta \eta^2 \frac{dh}{d\eta} = \frac{1}{3\pi},$$

and yields the following results for the logarithm of the partition function and the entropy,

$$S_I = 4 \ln Z_I = \frac{\mathcal{A}}{\ell^2} \frac{\pi^2 (\tilde{T} \tilde{L})^3}{45} \underset{2\pi \tilde{T} \tilde{L}=1}{=} \frac{\mathcal{A}/4\ell^2}{90\pi}, \quad (55)$$

in agreement with [1], [2] provided ℓ is identified with the Planck length $\ell_P = \sqrt{\hbar G}$.

We start the discussion of our results by emphasizing the difference between the thermodynamic and the entanglement partition functions and entropies. In 4-dimensional Minkowski space these thermodynamic quantities of a scalar massless field are proportional to \hbar^{-3} while the entanglement quantities in Rindler space or in curved spaces and close to a nonextremal horizon are independent of \hbar , cf. Eqs. (14, 15, 49). This difference reflects the difference in the number densities of the massless particle of the thermodynamic $(e^{\hbar\omega/T} - 1)^{-1}$ and of the entanglement $(e^{2\pi\omega/\kappa} - 1)^{-1}$ entropy (cf. Eqs. (6, 7)). The absence of \hbar suggests a correspondence between the entanglement partition function and entropy and the thermodynamic quantities of a classical field at finite temperature in an appropriately chosen background, cf. [24].

Of interest is also the comparison of two different approximate analytical methods to calculate the entanglement entropy of Eqs. (53) and (55) which exhibit very different dependences on size \tilde{L} and “temperature” \tilde{T} . While, due to the dominance by the lowest state, the expression (53) reflects the involved dispersion relation, the expression (55) exhibits the simple volume and temperature dependence of the entropy of a massless scalar field at finite temperature in 4-dimensional Minkowski space. This similarity in the structure is due to the replacement of the sum by an integration which would be justified only in the high “temperature” limit $\tilde{\beta}\omega \ll 1$ while the observed dominance of the $n = 1$ state shows that the “low temperature” limit is actually realized as far as the excitation of modes propagating perpendicular to the horizon is concerned. This qualitative difference in the structure of the thermodynamic quantities results in large differences of the values (53) and (55) of partition function and entropy by a factor of 233 and 115 respectively. Up to 1% this large overestimate actually arises from the integration (54) over the interval $0 \leq \nu \leq 1.6$ (cf. Fig. 4), i.e. where effectively only the modes $n = 1, 2$ contribute.

The derivative of the entropy with respect to $\tilde{\beta}$ yields the heat capacity (cf. [25] for calculation of C in the brick wall model),

$$C = -\tilde{\beta} \frac{\partial}{\partial \tilde{\beta}} S = \kappa \frac{\partial}{\partial \kappa} \left(\frac{\mathcal{A}}{4\ell^2} s(m_\ell) \right) = s(m_\ell) \kappa \frac{\partial}{\partial \kappa} \frac{\mathcal{A}}{4\ell^2},$$

where we have used the fact that the dimensionless quantity $s(m_\ell)$ cannot depend on κ , the only dimensionful quantity. Unlike in the definition of the entropy (49), where for technical reasons we have treated $\tilde{\beta}$ and $2\pi/\kappa$ as 2 independent variables, here we have to identify from the beginning these two quantities since the change in $\tilde{\beta}$ is assumed to be induced by a change in the mass and/or other parameters (cf. section II C) of the black hole. Thus entropy and heat capacity of the quantum field are simply related to the corresponding quantities of the underlying black hole

$$C = s(m_\ell) C_{BH}, \quad S = s(m_\ell) S_{BH}, \quad (56)$$

with the entropy, S_{BH} , and heat capacity, C_{BH} , of the black hole. The values of the area \mathcal{A} of various spaces are given by Eqs. (48), (43), (45), (46) and (47).

Other quantities of interest are the internal (U) and the free (F) energies which, in analogy with thermodynamics, are defined as

$$U = -\frac{\hbar\kappa}{2\pi} \kappa \partial_\kappa \ln Z, \quad F = U - \frac{\hbar\kappa}{2\pi} S.$$

Approximately, the internal energy for a Schwarzschild black hole of mass M is given by (cf. Eqs. (45, 53))

$$U \approx \frac{\hbar G M}{8\ell^2} S \underset{\ell=\ell_P}{=} \frac{M}{8} S \approx M \cdot 10^{-5}.$$

The accuracy of the results of the brick wall model is hard to assess. For instance, the choice of the Planck length, as the distance of the brick-wall to the horizon is, within certain limits, arbitrary. Also the properties of the cutoff procedure affect the value of $s(m_\ell)$. While the choice of the boundary condition which, if many modes contribute, is irrelevant [4], it becomes important if only one or a few modes dominate. We demonstrate this uncertainty by modifying the brick wall model and replace the Dirichlet (Eq. (10)) by the Neumann boundary condition which in PD amounts to replace in Eq. (17) $n\pi/\omega \rightarrow (n - 1/2)\pi/\omega$ resulting in the following expression for the partition function,

$$\ln \tilde{Z}_{Ne} = \frac{\tilde{\beta}\mathcal{A}}{\pi\ell^2} \int_0^\infty \frac{e^{(2\arg \Gamma(1+i\omega)+\pi)/\omega} d\omega}{(e^{\tilde{\beta}\omega} - 1)(e^{2\pi/\omega} - 1)}. \quad (57)$$

Numerical evaluation of this expression yields, in comparison with the results (PD) of Table II an enhancement of the partition function by a factor of 32 and of the entropy by a factor of 22.

To complete our considerations concerning the thermodynamic quantities it remains to consider, for spaces with spherical symmetry, the $l = 0$ contribution to the partition function. As we have seen (cf. Fig. 3), this contribution to the partition function cannot be approximated by the near horizon approximation. In the case of de Sitter space, the $l = 0$ contribution can be determined analytically. With the help of Eq. (38) we obtain,

$$\ln Z_{l=0} = - \sum_{n=1}^{\infty} \ln \left(1 - \exp \left(\frac{(2n-1)\pi^2}{\xi_0 + 1 - \ln 2} \right) \right) = 1.32,$$

which is larger than $\sum_n \zeta_n$ (first line of Table II) by a factor of $4.1 \cdot 10^5$. The relevance of this contribution depends on the density of states which is 1 for $l = 0$ and $\mathcal{A}/4\ell^2$ (cf. Eq. (14)) for large l .

B. Effects of masses on thermodynamic properties

With introduction of a mass, a new scale enters, (for a calculation of the free energy for massive fields within the WKB approximation cf. [26]). The distance ℓ to the horizon appears not only as prefactor ℓ^{-2} in the partition function (cf. Eq. (14)). According to Eq. (15) the thermodynamic properties of both massive and massless fields are determined by the same quantities, $\mathcal{K}_n(\omega)$. Differences result exclusively from the surface terms, $\sim \mathcal{K}_n^2$, and the

presence of the non-vanishing lower limit ω_n^0 in the ω integration which is determined by m_ℓ , the product of the mass and the distance to the horizon cf. Eq. (16). As indicated by Fig. 4, the effect of the non-vanishing surface term and of the lower limit of integration for $n = 1$ is, at the level of 1% or smaller, negligible for $\omega \leq 0.5$. Therefore the minimal mass, necessary for affecting the thermodynamic quantities, must satisfy (cf. (Eq. (16))),

$$m \geq \mathcal{K}_1(0.5) \ell^{-1} = 2\pi \mathcal{K}_1(0.5) T_\ell$$

where T_ℓ denotes the Tolman temperature at the distance ℓ from the horizon. Identifying ℓ with the Planck length the above inequality reads in terms of the Planck mass M_P

$$m \geq 2 \cdot 10^{-3} M_P. \quad (58)$$

Thus unless there are particles with masses of the order of at least $10^{-3} \times$ the Planck mass (“Micro black holes”), or there is a reason to increase the distance of the boundary to the horizon from the Planck length to at least $10^{-3} \times$ Compton wavelength of the corresponding particle, the mass of the particle does not affect the thermodynamic quantities.

Given the independence of the thermodynamic quantities from the mass of the particles outside the range (58) we can get a rough estimate of the entropy generated by the particles of the standard model. To this end, we assume that, apart from the multiplicity, all fundamental particles (leptons, quarks, gauge bosons and the Higgs particle) contribute the same amount to the entropy resulting in a value of $s(0)$ (cf. Table II) of the order of 10^{-2} . The ambiguity in the choice of the boundary condition results in an uncertainty of a factor of 22 (cf. Eq. (57)). Together with the uncertainty in choosing value of the distance to the horizon, this estimate could be wrong by one or two orders of magnitude. Beyond this uncertainty we also have to take into account that all particles beyond the standard model with masses, in the range,

$$1 \text{ TeV} \leq m \leq 10^{13} \text{ TeV},$$

for instance, possible supersymmetric partners of the particles in the standard model, contribute with the same weight to the partition function and other thermodynamic quantities as the particles with masses below 1 TeV.

Without specifying the origin, we now study the effect of sufficiently large masses on thermodynamic quantities in Rindler space and compare the results with an appropriately chosen system in Minkowski space. This system is assumed, as above, to be enclosed in a finite region with extension \tilde{L} in the transverse direction (cf. Eq. (13)) with periodic boundary conditions imposed. In the longitudinal direction we require vanishing of the fields at the two endpoints of a finite interval of size λ . A standard calculation of the partition function yields,

$$\ln Z_M = \tilde{L}^2 \ln z_M = \tilde{L}^2 \sum_{n=1}^{\infty} \zeta_n^M(\tilde{\beta}, \omega_n^M), \quad \zeta_n^M(\tilde{\beta}, \omega_n^M) = -\frac{1}{2\pi} \int_{\omega_n^M}^{\infty} \omega d\omega \ln(1 - e^{-\tilde{\beta}\omega}) \quad (59)$$

with

$$\omega_n^M = \sqrt{m_M^2 + n^2 \pi^2 / \lambda^2}.$$

Dimensionful quantities are given in units of appropriate powers of κ . Furthermore we choose $\tilde{\beta} = 2\pi$ and determine the value of λ such that, for $m_M = 0$ (cf. Eq. (15)), $\zeta_1(2\pi, 0) = \zeta_1^M(2\pi, \pi/\lambda)$, which is satisfied if $\lambda = 2.11$, and implies $\omega_1^M = 1.49$. With this choice and for

vanishing mass m , the sums in Eqs.(59) are dominated by the $n = 1$ terms and the partition function of the Minkowski and Rindler spaces are identical by construction, while the values of the entropy differ by 14%.

The results for partition function and entropy as a function of the mass $m_\ell = m_P$ (14) for $\ell = \ell_P$ are shown in Fig. 5. In order to compare properly the Rindler (14) and Minkowski

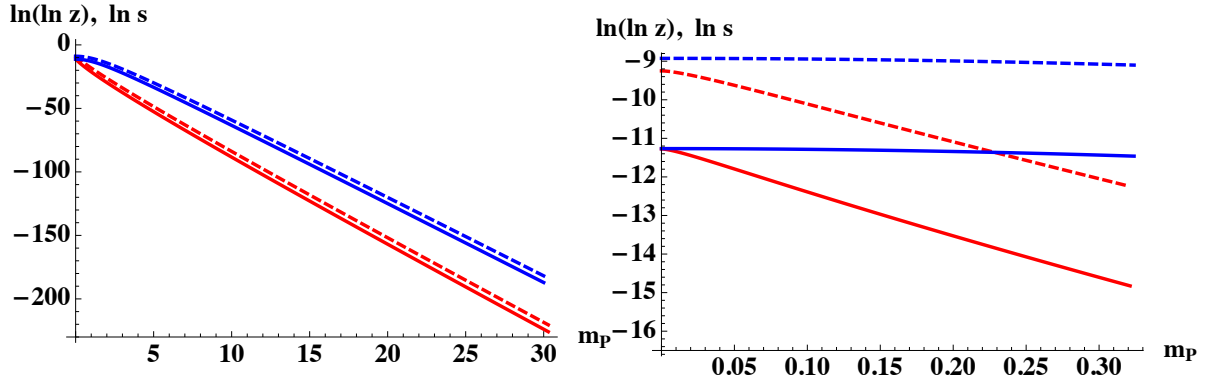


FIG. 5. Logarithm of the partition function $\ln z$ (solid lines) and of the entropy s (dashed lines) as function of m_P (red curves) at temperature $1/2\pi$ in comparison with the same quantities $\ln z_M$ and s_M (blue curves) in Minkowski space.

(59) space partition functions we consider the case $\xi_0 = 0$ which makes the prefactors of these partition functions equal. With the choice of λ , at $m = 0$ the values of the partition functions $\ln Z$ in Rindler and Minkowski space coincide. The difference of the two partition functions increases monotonically and at $m = 30$ they differ by a factor of 10^{17} . Two mechanisms are responsible for the differences in the two partition functions. The rapid decrease of the Rindler space partition function by a factor of 30 in the interval $m \leq 0.3$ is accompanied by a decrease of just 10% of the Minkowski space partition function. This different behavior reflects the unusual property of the Rindler or de Sitter space dispersion relations at small \mathcal{K} where minor changes of $\ln \omega$ induce large changes in $\ln \mathcal{K}$ as displayed in Fig. 1. In the asymptotic regime $\beta \omega_1^0 \gg 1$ the suppression of the partition function and entropy in Rindler as compared to Minkowski space is a result of the difference in the density of states (cf. Eq. (22)). Of the same origin is also the difference in the contribution from states with $n > 1$ which, for $m = 30$, accounts for 40% of the Minkowski space partition function and is negligible in Rindler space (cf. Eq. (23)).

IV. CONCLUSIONS

Dispersion relations have been shown to be useful tools for understanding the kinematics of scalar fields in space-times with a horizon. In particular, in a large range of the kinematics, they exhibit properties which, after choosing appropriate scales, are universal, i.e., independent of the details of the space times, as our explicit comparison of dispersion relations of Rindler and de Sitter space demonstrates. Dispersion relations provide a direct avenue to approximate analytical and numerical computation of the density of states, the essential ingredient for the thermodynamics of quantum fields in spaces with a horizon. The central results for the thermodynamic quantities is summarized in expression (53) and

Table II which imply that up to a correction of 0.3%, partition function and entropy are generated by a single mode controlling the motion perpendicular to the horizon. As shown in both the exact numerical and the approximate analytical evaluations, the validity of the semiclassical approximation and the assumption of a large number of modes contributing to the thermodynamic quantities cannot be sustained. Revisions are necessary whenever the semiclassical approximation has been used in the calculation of thermodynamic quantities.

New insights into the dynamics of quantum fields of higher spin, in particular of photons, cf. [27], [28], [29] and gravitons via dispersion relations can be expected. The imaginary parts of the corresponding stationary propagators [17], closely related to the dispersion relations, indicate significant differences between fields of different spin. Also the application to quantum fields in rotating black holes [30], [31], [32] promises to introduce a new element in the role of the probably complex dispersion relation. With the one mode dominance of the thermodynamic quantities, a hidden “parameter” specifying the boundary condition, emerges which, as we have seen, (cf. Eq. (57)), influences severely partition function and entropy and needs to be determined.

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